ON THE HIGHER RANK NUMERICAL RANGE OF THE SHIFT OPERATOR

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ABSTRACT. For any n-by-n complex matrix T and any $1 \leqslant k \leqslant n$, let $\Lambda_k(T)$ the set of all $\lambda \in \mathbb{C}$ such that $PTP = \lambda P$ for some rank-k orthogonal projection P be its higher rank-k numerical range. It is shown that if S_n is the n-dimensional shift on $\mathbb{C}^n$ then its rank-k numerical range is the circular disc centred in zero and with radius $\cos\dfrac{k\pi}{n+1}$ if $1 < k \leqslant \left[\frac{n+1}{2}\right]$ and the empty set if $\left[\frac{n+1}{2}\right] < k \leqslant n$, where $[x]$ denote the integer part of x. This extends and rafines previous results of U. Haagerup, P. de la Harpe [8] on the classical numerical range of the n-dimensional shift on $\mathbb{C}^n$. An interesting result for higher rank-k numerical range of nilpotent operator is also established.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the collection of all bounded linear operator on $\mathcal{H}$. The numerical range of an operators T in $\mathcal{B}(\mathcal{H})$ is the subset

$$W(T) = \{< Tx, x > \in \mathbb{C}; x \in \mathcal{H}, \|x\| \leqslant 1\}$$

of the plane, where $< ., . >$ denotes the inner product in $\mathcal{H}$ and the numerical range of T is defined by

$$\omega_2(T) = \sup\{|z|; z \in W(T)\}.$$

We denote by S the unilateral shift acting on the Hardy space $\mathbb{H}^2$ of the square summable analytic functions.

$$\begin{array}{cccc} S & : & \mathbb{H}^2 & \to & \mathbb{H}^2 \\ & & f & \mapsto & zf(z) \end{array}$$

Beurling's theorem implies that the non zero invariant subspaces of S are of the forme $\phi\,\mathbb{H}^2$, where ϕ is some inner function . Let $S(\phi)$ denote the compression of S to the space $H(\phi) = \mathbb{H}^2 \ominus \phi\,\mathbb{H}^2$:

$$S(\phi)f(z) = P(zf(z)),$$

where P denotes the ortogonal projection from $\mathbb{H}^2$ onto $H(\phi)$. The space $H(\phi)$ is a finite-dimensional exactly when ϕ is a finite Blaschke product. The numerical radius and numerical range of the model operator $S(\phi)$ seems to be important and have many applications. In [1], Badea and Cassier showed that there is relationship between numerical radius of $S(\phi)$ and Taylor coefficients of positive rational functions on the torus and more recently in [6], the author gave an extension of this result. However the evaluation of the numerical radius of $S(\phi)$ under an explicit form is always an open problem. The reader may consult [6] for an estimate of $S(\phi)$

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where ϕ is a finite Blashke product with unique zero. In the particular case where $\phi(z) = z^n$, $S(\phi)$ is unitarily equivalent to S_n where

$$S_n = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

In [8]; it is proved that $W(S_n)$ is the closed disc $D_n = \left\{ z \in \mathbb{C}; |z| \leqslant \cos \frac{\pi}{n+1} \right\}$ and $\omega_2(S_n) = \cos \frac{\pi}{n+1}$ and more general

Theorem 1.1 ([8]). *Let T be an operator on $\mathcal{H}$ such that $T^n = 0$ for some $n \geq 2$. One has:*

$$\omega_2(T) \leqslant \|T\| \cos \frac{\pi}{n+1}$$

and $\omega_2(T) = \|T\| \cos \frac{\pi}{n+1}$ when T is unitarily equivalent to $\|T\| S_n$.

In this mathematical note, we extend this result to the higher rank-k numerical range of the shift. The notion of the the higher rank-k numerical range of $T \in \mathcal{B}(\mathcal{H})$ is introduced in [4] and it's denoted by:

$$\Lambda_k(T) = \{\lambda \in \mathbb{C} : PTP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P\},$$

The introduction of this notion was motivated by a problem in quantum error correction; see [5]. If P is a rank-1 orthogonal projection then $P = x \otimes x$ for some $x \in \mathbb{C}^n$ and $PTP = < Tx, x > P$. Then when $k = 1$, this concept is reduces to the classical numerical range $W(T)$, which is well known to be convex by the Toeplitz-Hausdorff theorem; for exemple see [10] for a simple proof. In [2], it's conjectured that $\Lambda_k(T)$ is convex, and reduced the convexity problem to the problem of showing that $0 \in \Lambda_k(T')$ where

$$T' = \begin{pmatrix} I_k & X \\ Y & -I_k \end{pmatrix}$$

for arbitrary $X, Y \in \mathcal{M}_k$ (the algebra of $k \times k$ complex matrix). They further reduced this problem to the existance of a Hermitian matrix H satisfying the matrix equation

$$I_k + MH + HM^* - HRH = H \tag{1.1}$$

for arbitrary $M \in \mathcal{M}_k$ and a positive definite $R \in \mathcal{M}_k$. In [16], H. Woerdeman proved that equation (1.1) is equivalent to Ricatti equation:

$$HRH - H(M^* - I_k/2) - (M - I_k/2)H - I_k = 0_k, \tag{1.2}$$

and using the theory of Ricatti equations (see [9], Theorem 4), the equation (1.2) is solvable which prove the convexity of $\Lambda_k(T)$. In [4], the authors showed that if $\dim\mathcal{H} < \infty$ and $T \in \mathcal{B}(\mathcal{H})$ is a Hermitian matrix with eigenvalues $\lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_n$ then the rank-k nuemrical range $\Lambda_k(T)$ coincides with $[\lambda_k, \lambda_{n+1-k}]$ which is a non-degenerate closed interval if $\lambda_k < \lambda_{n+1-k}$, a singleton set if $\lambda_k = \lambda_{n+1-k}$ and an empty set if $\lambda_k > \lambda_{n+1-k}$. In [13], the authors proved that if $\dim\mathcal{H} = n$

$$\Lambda_k(T) = \bigcap_{\theta \in [0, 2\pi[} \left\{ \mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k \left(e^{i\theta}T + e^{-i\theta}T^* \right) \right\},$$

for $1 \leqslant k \leqslant n$, where $\lambda_k(H)$ denote the kth largest eigenvalue of the hermitian matrix $H \in \mathcal{M}_n$. This result establishes that if $\dim\mathcal{H} = n$ and $T \in \mathcal{B}(\mathcal{H})$ is a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$\Lambda_k(T) = \bigcap_{1 \leqslant j_1 < \cdots < j_{n-k+1} \leqslant n} \operatorname{conv}\left\{\lambda_{j_1}, \dots, \lambda_{j_{n+1-k}}\right\}.$$

We close this section by the following properties wich are easly checked. The reader may consult [2],[3],[4],[5],[7] and [11].

P1. For any a and $b \in \mathbb{C}$, $\Lambda_k(aT + bI) = a\Lambda_k(T) + b$.
P2. $\Lambda_k(T^*) = \overline{\Lambda_k(T)}$.
P3. $\Lambda_k(T \oplus S) \supseteq \Lambda_k(T) \cup \Lambda_k(S)$.
P4. For any unitary $U \in \mathcal{B}(\mathcal{H})$, $\Lambda_k(U^*TU) = \Lambda_k(T)$.
P5. If T_0 is a compression of T on a subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\dim\mathcal{H}_0 \geq k$, then $\Lambda_k(T_0) \subseteq \Lambda_k(T)$.
P6. $W(T) \supseteq \Lambda_2(T) \supseteq \Lambda_3(T) \supseteq \dots$.

Some results from [1] will be also developed in this context in a forthcoming paper.

2. MAIN THEOREM

In the following theorem we give the higher rank-k numerical range of the n-dimensional shift on $\mathbb{C}^n$.

Theorem 2.1. *For any $n \geq 2$ and $1 \leqslant k \leqslant n$, $\Lambda_k(S_n)$ coincides with the circular disc $\{z \in \mathbb{C} : |z| \leqslant \cos \dfrac{k\pi}{n+1}\}$ if $1 \leqslant k \leqslant \left[\frac{n+1}{2}\right]$ and the empty set if $\left[\frac{n+1}{2}\right] < k \leqslant n$.*

Proof. First observe that

$$\begin{aligned}
\Lambda_k(S_n) &= \bigcap_{\theta \in [0,2\pi[} \left\{\mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k\left(e^{i\theta}S_n + e^{-i\theta}S_n^*\right)\right\} \\
&= \bigcap_{\theta \in [0,2\pi[} \left\{\mu \in \mathbb{C} : Re(e^{i\theta}\mu) \leqslant \frac{1}{2}\lambda_k\left(e^{i\theta}S_n + e^{-i\theta}S_n^*\right)\right\} \\
&= \bigcap_{\theta \in [0,2\pi[} e^{i\theta}\left\{z \in \mathbb{C} : Re(z) \leqslant \frac{1}{2}\lambda_k\left(e^{i\theta}S_n + e^{-i\theta}S_n^*\right)\right\} \qquad (2.1)
\end{aligned}$$

On the other hand, we have

$$e^{i\theta}S_n + e^{-i\theta}S_n^* = \begin{pmatrix}
0 & e^{-i\theta} & 0 & \dots & 0 & 0\dots \\
e^{i\theta} & 0 & e^{-i\theta} & \dots & 0 & 0\dots \\
0 & e^{i\theta} & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & e^{-i\theta} \\
0 & 0 & 0 & \dots & e^{i\theta} & 0
\end{pmatrix}.$$

Note that $e^{i\theta}S_n + e^{-i\theta}S_n^*$ is a Toeplitz matrix associated to the Toeplitz form

$$f_\theta(t) = 2\cos(\theta + t).$$

The eigenvalues satisfy the caracteristic equation

$$\begin{aligned}\Delta_n(\lambda) &= Det\left(e^{i\theta}S_n + e^{-i\theta}S_n^*\right)\\ &= \begin{vmatrix} -\lambda & e^{-i\theta} & 0 & \dots & 0 & 0\dots \\ e^{i\theta} & -\lambda & e^{-i\theta} & \dots & 0 & 0\dots \\ 0 & e^{i\theta} & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda & e^{-i\theta} \\ 0 & 0 & 0 & \dots & e^{i\theta} & -\lambda \end{vmatrix}\end{aligned}$$

Expanding this determinant, we obtain the recurrence relation

$$\Delta_n(\lambda) = -\lambda\Delta_{n-1} - \Delta_{n-2},\ n = 2, 3, 4, \dots,$$

This recurrence relation holds also for $n = 1$ provided we put $\Delta_0 = 1$ and $\Delta_{-1} = 0$. In order to find an explicit representation of $\Delta_n(\lambda)$, we write convenently

$$\lambda = 2\cos(\theta + t) = f_\theta(t)$$

and form the caracteristic equation

$$\rho^2 = -\lambda\rho - 1 = -2\rho\cos(\theta + t) - 1$$

with the roots $-e^{i(\theta+t)}$ and $-e^{-i(\theta+t)}$ so that

$$\Delta_n(2\cos(\theta + t)) = (-1)^n(Ae^{in(\theta+t)} + Be^{-in(\theta+t)})$$

where the constants A and B can be determined from the cases $n = -1$ and $n = 0$. Thus

$$\Delta_n(2\cos(\theta + t)) = (-1)^n \frac{\sin((n + 1)(\theta + t))}{\sin(\theta + t).}$$

This yields the eigenvalues

$$\lambda_\nu = 2\cos(\frac{\nu\pi}{n + 1}),\ \nu = 1, 2, \dots n.$$

This implies of course that

$$\Lambda_k(S_n) = \bigcap_{\theta\in[0,2\pi[} e^{i\theta}\left\{z \in \mathbb{C} : Re(z) \leqslant \cos(\frac{k\pi}{n + 1})\right\}$$

Thus $\Lambda_k(S_n)$ is the intersection of closed half planes. We note that $\cos(\frac{k\pi}{n+1})$ is positive if and only if $k \leqslant \left[\frac{n+1}{2}\right]$.

Case 1. If $k \leqslant \left[\frac{n+1}{2}\right]$ In this case $\Lambda_k(S_n)$ is circular disc $\{z \in \mathbb{C} : |z| \leqslant \cos\frac{k\pi}{n + 1}\}$.

Case 2. If $k > \left[\frac{n+1}{2}\right]$, then

$$\begin{aligned}\Lambda_k(S_n) &\subseteq \left\{z \in \mathbb{C} : Re(z) \leqslant \cos(\frac{k\pi}{n + 1})\right\} \bigcap e^{i\pi}\left\{z \in \mathbb{C} : Re(z) \leqslant \cos(\frac{k\pi}{n + 1})\right\}\\ &= \emptyset.\end{aligned}$$

This completes the proof. □

Theorem 2.2. For any integer $k \geq 1$

$$\Lambda_k(S) = D(0, 1)$$

Proof. Let a fixed $k \geq 1$, we have $D(0,1) \subseteq \Lambda_k(S)$ which is du to (P.5) and theorem (2.1). Now let λ in $\Lambda_k(S)$ then there exists a rank-k orthogonal projection P such that $PSP = \lambda P$. Let denote by U_θ the unitary operator on $\mathbb{H}^2$ defined by $U_\theta(f)(z) = f(ze^{-i\theta})$, then if we denote by Q the rank-k orthogonal projection $U_\theta P U_\theta^*$ we can easly check that $QSQ = \lambda e^{i\theta}$ which implies that $\Lambda_k(S)$ is a circular disc centred in 0. On the other hand if $1 \in \Lambda_k(S)$ then $1 \in W(S)$ and there exists a unitary $f \in \mathbb{H}^2$ such that $< Sf, f >= 1$ wich implies that 1 is un eigenvalue for S which is absurd. □

On the sequel of this paper, let denote by

$$\rho(k,r) = \begin{cases} k/r & \text{if } k/r \text{ is is integer} \\ [k/r]+1 & \text{unless} \end{cases}$$

where k and r are arbitrary numbers.

***Lemma* 2.3.** *For a fixed $n \geq 1$ and $r \geq 1$, let denote by $\lambda_1 > \cdots > \lambda_n$; n real numbers and $(\lambda'_p)_{1\leqslant p\leqslant nr}$ a finite sequence defined by:*

$$\lambda'_1 = \cdots = \lambda'_r = \lambda_1, \ldots, \lambda'_{(n-1)r+1} = \cdots = \lambda'_{nr} = \lambda_n.$$

Then for each $1 \leqslant k \leqslant nr$, the kth largest term of $(\lambda'_t)_{1\leqslant t\leqslant nr}$ is $\lambda_{\rho(k,r)}$.

Proof. The claim is obvious in the case where $r = 1$. We may assume $r \geq 2$. We prove the result by induction on k. If $k = 1$, then the largest term is $\lambda_1 = \lambda_{\rho(1,r)}$. So the result hold for $k = 1$. Assume that $k > 1$, and the reslut is valid for the mth largest term of $(\lambda'_t)_{1\leqslant t\leqslant nr}$ whenever $m < k$.

Case 1. Suppose that $\rho(k-1,r) = \frac{k-1}{r}$, then there exists $1 \leqslant p \leqslant n-1$ such that $k-1 = pr$. By induction assumption, we have $\lambda_{\rho(k-1,r)} = \lambda'_{pr} = \lambda_p$, which implies that the kth largest term of $(\lambda'_t)_{1\leqslant t\leqslant nr}$ is

$$\lambda'_{pr+1} = \lambda_{p+1} = \lambda_{\frac{k-1}{r}+1} = \lambda_{[\frac{k}{r}]+1} = \lambda_{\rho(k,r)}.$$

Case 2. Suppose that $\rho(k-1,r) = [\frac{k-1}{r}]+1$, then there exist $1 \leqslant q \leqslant n-1$ and $1 \leqslant s \leqslant r-1$ such that $k-1 = qr+s$. First, note that $\rho(k-1,r) = \rho(k,r)$. On the other hand, by induction assumption, we have $\lambda_{\rho(k-1,r)} = \lambda'_{qr+s} = \lambda_{q+1}$. Consequently the kth largest term of $(\lambda'_t)_{1\leqslant t\leqslant nr}$ is

$$\lambda'_{qr+s+1} = \lambda_{q+1} = \lambda_{\rho(k-1,r)} = \lambda_{\rho(k,r)}.$$

The proof is now complete. □

Let $D_T = (I_N - T^*T)^{1/2}$ be the defect operator of T and $\mathcal{D}_T$ the closed range of D_T. Let denote by $r = \dim\mathcal{D}_T$.

***Theorem* 2.4.** *Consider $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| \leqslant 1$ and $T^n = 0$. Then $\Lambda_k(T)$ is contained in the circular disc $\{z \in \mathbb{C} : |z| \leqslant \cos(\frac{\rho(k,r)\pi}{n+1})\}$ if $1 \leqslant \rho(k,r) \leqslant [\frac{n+1}{2}]$ and empty if $\rho(k,r) > [\frac{n+1}{2}]$.*

Proof. If T is a contaction with $T^n = 0$, then T can be viewed as a compression of $I_r \otimes S_n^*$ acting on the Hilbert space $\mathcal{D}_T \otimes \mathbb{C}^n$. Consider the isometry $\mathcal{H} \to \mathcal{D}_T \otimes \mathbb{C}^n$,

$$V(x) = \sum_{t=1}^{n} D_T T^{t-1} x \otimes e_t$$

where $\{e_l\}_{l=1}^{n}$ is the canonical basis of $\mathbb{C}^n$. Note that

$$VTx = \sum_{t=1}^{n} D_T T^t x \otimes e_t = \sum_{t=1}^{n-1} D_T T^t x \otimes e_t = (I_r \otimes S_n^*)Vx.$$

It follows that

$$T = V^*(I_r \otimes S_n^*)V$$

and from (P.5)

$$\Lambda_k(T) = \Lambda_k(V^*(I_r \otimes S_n^*)V) \subseteq \Lambda_k(I_r \otimes S_n^*), \text{ for any } 1 \leqslant k \leqslant nr. \tag{2.2}$$

Now,

$$\begin{aligned}
&\Lambda_k(I_r \otimes S_n^*)\\
&= \bigcap_{\theta\in[0,2\pi[} \left\{\mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k\left(e^{i\theta}(I_r \otimes S_n^*) + e^{-i\theta}(I_r \otimes S_n^*)^*\right)\right\}\\
&= \bigcap_{\theta\in[0,2\pi[} \left\{\mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k\left(e^{i\theta}(I_r \otimes S_n^*) + e^{-i\theta}(I_r \otimes S_n)\right)\right\}\\
&= \bigcap_{\theta\in[0,2\pi[} \left\{\mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k\left(I_r \otimes (e^{i\theta}S_n + e^{-i\theta}S_n^*)\right)\right\}\\
&= \bigcap_{\theta\in[0,2\pi[} \left\{\mu \in \mathbb{C} : e^{i\theta}\mu + e^{-i\theta}\overline{\mu} \leqslant \lambda_k\left(\oplus_i^r(e^{i\theta}S_n + e^{-i\theta}S_n^*)\right)\right\}\\
&= \bigcap_{\theta\in[0,2\pi[} e^{i\theta}\left\{z \in \mathbb{C} : Re(z) \leqslant \cos(\frac{\rho(k,r)\pi}{n+1})\right\}
\end{aligned}$$

where the last equality is due to the lemma (2.2) and theorem (2.1). Thus

$$\Lambda_k(I_r \otimes S_n^*) = \begin{cases} \overline{D(0, \cos(\frac{\rho(k,r)\pi}{n+1}))} & \text{if } 1 \leqslant \rho(k,r) \leqslant [\frac{n+1}{2}] \\ \emptyset & \text{if } [\frac{n+1}{2}] < \rho(k,r) \leqslant n \end{cases}$$

Therefore,
if $1 \leqslant k \leqslant nr$, (2.2) implies that $\Lambda_k(T) \subseteq \overline{D(0, \cos(\frac{\rho(k,r)\pi}{n+1}))}$ if $1 \leqslant \rho(k,r) \leqslant [\frac{n+1}{2}]$ and empty if $[\frac{n+1}{2}] < \rho(k,r) \leqslant n$. Finally, if $k > nr$, $\Lambda_k(T) = \emptyset$ from (P6). □

Corollary 2.5 (U. Haagerup, P. de la Harpe,[8]). *Consider $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| \leqslant 1$ and $T^n = 0$. Then we have $\omega_2(T) \leqslant \cos(\frac{\pi}{n+1})$.*

Proof. $T = V^*(I_r \otimes S_n^*)V$ where $V : H \to \mathcal{D}_T \otimes \mathbb{C}^n$,

$$V(x) = \sum_{t=1}^{n} D_T T^{t-1} x \otimes e_t.$$

Now

$$W(T) = \Lambda_1(T) = \Lambda_1(V^*(I_r \otimes S_n)V) \subseteq \Lambda_1(I_r \otimes S_n) = \overline{D(0, \cos\frac{\pi}{n+1})}.$$

□

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